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A COUNTER-EXAMPLE AND CORRECTION TO A THEOREM OF VENTER. (U)

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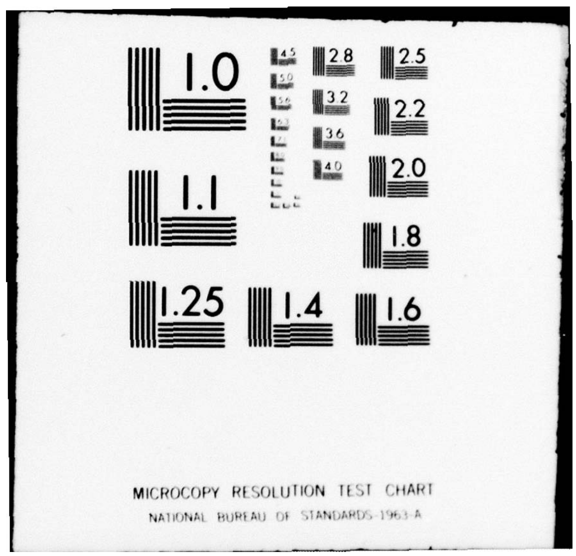
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(6) A COUNTER-EXAMPLE AND CORRECTION TO A THEOREM OF VENTER.

by

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A Counter-example and Correction to a Theorem of Venter

Let H be a set and $(T_n, n = 1,2,\ldots)$ a sequence of transformations of H into itself. Let X_1 and (U_n) be random elements in H and generate the sequence (X_n) by

$$X_{n+1} = T_n(X_n) + U_n .$$

Theorems giving conditions under which (X_n) is "stochastically attracted" towards a given subset of H and will eventually be within or arbitrarily close to this set (in some sense) are called Dvoretzky stochastic approximation theorems.

In this note we point out that one such theorem due to Venter (1966) is erroneous by giving a counter-example. We also rectify this by strengthening one of the conditions required by Venter. For the sake of easy reference we quote the theorem under discussion (Theorem 3 in Venter (1966)).

Theorem:

Let H be a real separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $\mathcal{X}$ be the σ-field of subsets of H generated by the open sets. Let $(\Omega, \mathcal{A}, P)$ be a probability measure space; the elements of Ω are generically denoted by ω. Let S_n be a transformation of $H\times\Omega$ into H. Let T_n be specified by

$$T_n(x_1,\ldots,x_n,\omega) = x_n - S_n(x_n,\omega)$$

and suppose that S_n satisfies the following conditions. For each $x \in H$ and $\omega \in \Omega_0 \in \mathcal{A}$ where $P(\Omega_0) = 1$,

$$\|S_n(x,\omega)\|^2 \leq \beta_n \|x - \theta\|^2 + \delta_n$$

for all n, where β_n, δ_n are non-negative real sequences such that

$$\Sigma\beta_n < \infty, \qquad \Sigma\delta_n < \infty .$$

Also for each $\epsilon > 0$, define

$$c_n(\epsilon,\omega) = \lim_{\epsilon \leq \|x-\theta\| \leq \epsilon^{-1}} 2(x-\theta, S_n(x,\omega))$$

and

(A) *suppose that there is a finite integer valued random variable N_ϵ such that for all $n > N_\epsilon(\omega)$ and for all $\omega \in \Omega_0$,*

(1) $$c_n(\epsilon,\omega) \geq \delta_n$$

while also

(2) $$\Sigma c_n(\epsilon,\omega) = \infty$$

(the italics are ours).

Define (X_n) by

$$X_1 \text{ is arbitrary with } EX_1^2 < \infty,$$

$$X_{n+1} = T_n(X_1(\omega),\ldots,X_n(\omega),\omega) + U_n(\omega)$$

where (U_n) is a sequence of random elements satisfying the conditions

(3) $$\Sigma\, E\|U_n\|^2 < \infty$$

and

(4) $$\Sigma\|E[U_n|\mathcal{B}_n]\| < \infty \text{ a.s.}$$

where $(\mathcal{B}_n)$ is an increasing sequence of sub σ-fields of $\mathcal{Q}$ having the properties that the random elements $\{X_1,\ldots,X_n,T(X_1),\ldots,T_n(X_1,\ldots,X_n)\}$ are measurable with respect to $\mathcal{B}_n$ for $n = 2,3\ldots$ and that

$$[\omega \in \Omega : n > N_\epsilon(\omega)] \in \mathcal{B}_n\ .$$

then $x_n \to \theta$ a.s. as $n \to \infty$. In addition if N_ϵ is degenerate and (U_n) satisfies (3) and instead of (4)

$$\Sigma(E\|E[U_n|\mathcal{B}_n]\|^2)^{1/2} < \infty .$$

Then $E\|X_n-\theta\|^2 \to 0$ as $n \to \infty$.

The theorem as stated is false as can be seen from the following example.

<u>Example</u>:

Let M be a mapping from R to R such that

$$M(x) = \begin{cases} x & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1\ . \end{cases}$$

Then $\quad x\,M(x) > 0 \quad$ for $\quad 0 \ne |x| \le 1$

and $|M(x)| \ge C|x|$ for $|x| \le 1$ with $0 < C \le 1$

X_1 be a fixed real number and define X_n for $n \geq 2$ recursively by

$$X_{n+1} = X_n - n^{-1}M(X_n).$$

Clearly if $|X_1| > 1$ then $X_n = X_1$ for all n. Thus for every $0 < a < C$, $n^a X_n \to \infty$ and $n^{2a} EX_n^2 = E(n^a X_n)^2 \to \infty$.

On the other hand as we shall see the process $Y_n = n^a X_n$ with

$$a = \min(1/2, C)$$

satisfies all the conditions of Venter's theorem.

Let

$$n^a X_n = Y_n .$$

Then

$$\begin{aligned} Y_{n+1} &= (n+1)^a X_n - (n+1)^a\, n^{-1}\, M(X_n) \\ &= (1+n^{-1})^a\, Y_n - (n+1)^a\, n^{-1}\, M(n^{-a} Y_n) . \end{aligned}$$

Thus using the notation of the theorem,

$$\begin{aligned} S_n(x,\omega) = S_n(x) &= (n+1)^a\, n^{-1}\, M(n^{-a}x) - [(1+n^{-1})^a - 1]x \\ &= (n+1)^a\, n^{-1}\, M(n^{-a}x) - [an^{-1} + O(n^{-2})]x . \end{aligned}$$

Hence

$$\begin{aligned} |S_n(x)| &\leq (n+1)^a\, n^{-1-a}\, |x| + [an^{-1} + O(n^{-2})]|x| \\ &\leq |x|[(1+a)n^{-1} + O(n^{-2})] \end{aligned}$$

so

$$\beta_n = 2(1+a)^2\, n^{-2} + O(n^{-3}) \text{ and}$$

$$\delta_n = 0 \quad .$$

Clearly $\Sigma\beta_n < \infty$ and $\Sigma\delta_n < \infty$. Further

$$2x\, S_n(x) = 2(n+1)^a\, n^{-1}\, |x|\, |M(n^{-a}x)| - 2[an^{-1} + O(n^{-2})]x^2 .$$

Let n be so large that $\epsilon^{-1}\, n^{-a} < 1$. Then for $\epsilon \leq |x| \leq \epsilon^{-1}$,

$$|M(n^{-a}x)| \geq C|xn^{-a}|$$

and therefore

$$\begin{aligned} 2x\, S_n(x) &= 2(1+n^{-1})^a Cn^{-1}\, |x|^2 - 2[an^{-1} + O(n^{-2})]x^2 \\ &= 2|x|^2\, n^{-1}[(1+n^{-1})^a C - a + O(n^{-2})] \\ &= 2|x|^2\, n^{-1}[C - a + O(n^{-2})] \quad . \end{aligned}$$

Thus $C_n(\epsilon,\omega) = C_n(\epsilon) = 2\epsilon^2\, n^{-1}[C - a + O(n^{-2})]$ and so

$$\Sigma\, C_n(\epsilon) = \infty$$

and

$C_n(\epsilon) \geq \delta_n$ if $n > N_\epsilon$ where N_ϵ is given by $(\epsilon N_\epsilon^a)^{-1} < 1$.

Thus all the conditions are satisfied but the conclusion is obviously wrong.

The error in Venter's proof is as follows: Venter uses a theorem, Theorem 1 in the same paper to prove Theorem 3 in his paper. Theorem 1 assumes (among other things) that

$$\|T_n(x_1,\ldots,x_n,\omega) - \theta\|^2 \leq \max[\alpha, (1+\beta_n)\|x_n - \theta\|^2 - \gamma_n] \quad \text{where}$$

$$\gamma_n(x_1,\ldots,x_n,\omega) \geq 0 \quad \text{if} \quad n > N(\omega). \tag{5}$$

But in the proof of Theorem 3, Venter shows that for any $\epsilon > 0$ and all (x_n) such that $\sup\|x_n\| < \epsilon^{-1}$, (5) holds. But then $N(\omega)$ depends on ϵ and Theorem 1 is not applicable.

The theorem can be corrected if Condition A is changed to read: "there exists a finite integer valued random variable N (instead of N_ϵ) such that (1) and (2) hold".

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Reference:

Venter, J. H. (1966). On Dvoretzky Stochastic Approximation Theorems. Ann. Math. Statist., 37, 1534-1544.

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In this note, ~~we point out~~ an error is pointed out in a result due to Venter on Dvoretzky stochastic approximation theorems by giving a counter-example. ~~We also~~ * rectified ~~this~~ by strengthening one of the conditions required by Venter.

* The error is

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